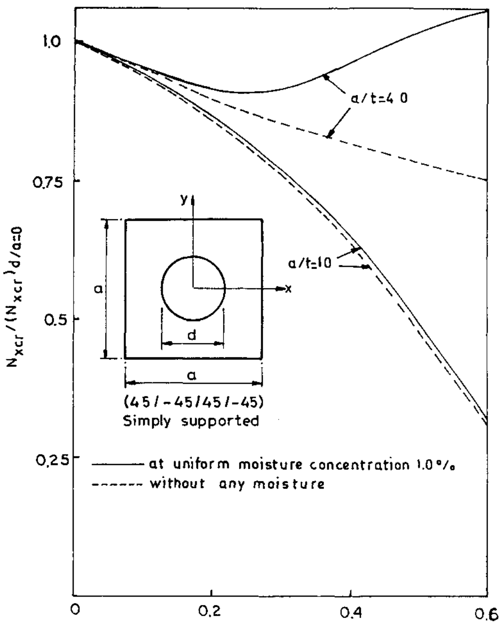



**Fig. 4 Effect of cutout size on the critical load (antisymmetric angle ply).**

of 40 are subjected to uniform moisture concentration of 1.0%. However, compared to the simply supported case, moisture presence has less effect on the fundamental frequency of clamped laminates with a cutout when $a/t$ is 40.

The effect of cutout size on the critical loads of simply supported symmetric and antisymmetric laminates, subjected to 1.0% uniform moisture concentration and in the absence of moisture, is shown in Figs. 3 and 4. In the case of laminates when $a/t$ is 10, the critical loads decrease with increase in cutout size. The critical loads of the laminates when $a/t$ is 40 also decrease with increase in cutout size in the absence of moisture; whereas, in the presence of 1.0% uniform moisture concentration, they first decrease ($d/a$ is about 0.4 for cross-ply laminates, $d/a$ are about 0.3 and 0.25 in respect of angle-ply symmetric and antisymmetric laminates, respectively) and then increase.

The critical loads of clamped symmetric and antisymmetric laminates when $a/t$ is 10 are reduced with increase in cutout size, but the initial reduction is sharp for cross-ply laminates. The critical loads of the clamped laminates when $a/t$ is 40 are also reduced with increase in cutout size in the absence of moisture; whereas in the presence of moisture, the critical loads decrease and/or increase as the case may be.

## Conclusions

The increase in the fundamental natural frequencies of thin laminates, with increase in cutout size, is greater in the presence of moisture. Moisture presence has negligible effect on the increase in the fundamental natural frequencies of thick laminates with increase in cutout size. The buckling behavior of thick laminates with a cutout is almost unchanged due to the presence of moisture. The critical loads of thin laminates with a cutout are very much affected in the presence of moisture.

## References

[1]Springer, G. (ed.), *Environmental Effects on Composite Materials*, Vol. 1, (1981), Vol. 2 (1984), Vol. 3 (1988), Technomic, Westport, CT.

[2]Ram, K. S. S., and Sinha, P. K., "Hygrothermal Effects on the Free Vibration of Laminated Composite Plates," *Journal of Sound and Vibration* (to be published).

[3]Ram, K. S. S., and Sinha, P. K., "Hygrothermal Effects on the Buckling of Laminated Composite Plates," *Composite Structures* (to be published).

[4]Chang, J. S., and Shio, F. J., "Thermal Buckling Analysis of Isotropic and Composite Plates with a Hole," *Journal of Thermal Stresses*, Vol. 13, No. 3, 1990, pp. 315-332.

[5]Reddy, J. N., "Large Amplitude Flexural Vibration of Layered Composite Plates with Cutouts," *Journal of Sound and Vibration*, Vol. 83, No. 1, 1982, pp. 1-10.

[6]Rajamani, A., and Prabhakaran, R., "Dynamic Response of Composite Plates with Cutouts, Part I: Simply Supported Plates, Part II: Clamped-Clamped Plates," *Journal of Sound and Vibration*, Vol. 54, No. 4, 1977, pp. 549-576.

[7]Nemeth, M. P., "Buckling Behavior of Compression Loaded Symmetrically Laminated Angle-Ply Plates with Holes," *AIAA Journal*, Vol. 26, No. 3, 1988, pp. 330-336.

[8]Tsai, S. W., and Hahn, H. T., *Introduction to Composite Materials*, Technomic, Westport, CT, 1980.

# Torsional Stiffness for Circular Orthotropic Beams

S. Dubigeon*
*Ecole Centrale de Nantes, 44072 Nantes, France*

## Nomenclature

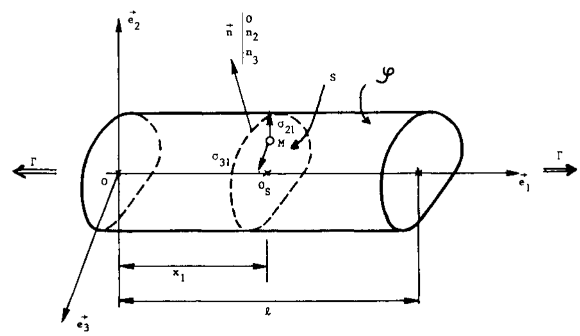
| | |
|---|---|
| $A$ | = tensor |
| $A{:}B$ | = double dot product: $A_{ij}B_{ji}$ |
| $A \cdot u$ | = simple dot product: $A_{ji}u_i e_j$ |
| $u$ | = vector |
| $u \cdot v$ | = simple dot product (inner product): $u_i v_i$ |
| $\nabla$ | = gradient operator: $\varphi\nabla = \varphi_{,i} e_i$ |
| $\cdot\nabla$ | = divergence operator: $u \cdot \nabla = u_{i,i}$, $A \cdot \nabla = A_{ij,j} e_i$ |
| $\wedge$ | = vector product or cross product |

## Introduction

FOR a long time, solutions for torsion of beam structures have been studied and solutions for many problems can be found in the literature; see Refs. 1-5, for example. Warping functions are known analytically for some problems, and either may be obtained using computers. But, in most of the cases the medium is isotropic. In this Note, we pose the problem for an orthotropic medium, and we give the solution for a circular section and propose some evident applications.

## Development

We consider a circular cylinder with axes in one of the directions of orthotropy of the medium, and we complete the basis by the two other directions of orthotropy. The body in equilibrium is only acted upon by a couple $\Gamma e_1$ on its base $x_1 = \ell$, and $-\Gamma e_1$ on its base $x_1 = 0$.




Received July 15, 1991; revision received Oct. 3, 1991; accepted for publication Oct. 21, 1991. 

*Professor, Structural Mechanics Laboratory, 1 rue de la Noë.

We use the approach of St.Venant, and we take $u_M$ as the displacement function.

$$u_M = \alpha(\mathbf{00}_s \wedge \mathbf{0}_S M + \varphi\ (x_2, x_3) e_1)$$

where $\alpha = \theta_{1,1}$ and $\varphi$ is the warping function. So

$$\gamma_{12} = \alpha(\varphi,_2 - x_3)$$

$$\gamma_{13} = \alpha(\varphi,_3 - x_2)$$

The state of stress must verify

$$\Sigma = \Sigma^T,\ \Sigma \cdot \nabla = \mathbf{0}$$

Then using the law of behavior,

$$\sigma_{13} = \sigma_{31} = G_{33}\gamma_{31} \qquad \text{and} \qquad \sigma_{12} = \sigma_{21} = G_{22}\gamma_{21}$$

We obtain

$$G_{22}\varphi,_{22} + G_{33}\varphi,_{33} = 0 \qquad \text{on } S$$

On the boundary of $S$, stress is equal to zero.

$$T = \Sigma \cdot n_\ell = \mathbf{0}$$

We obtain

$$G_{22}(\varphi,_2 - x_3)n_2 + G_{33}(\varphi,_3 + x_2)n_3 = 0 \qquad \text{on } \partial S$$

Results are written intrinsically

$$(G \cdot \varphi\nabla) \cdot \nabla = 0 \qquad \text{on } S$$

$$n \cdot G \cdot [\varphi\nabla + e_1 \wedge \mathbf{0}_S M] = 0 \qquad \text{on } \partial S$$

with $[G] =$ diag $[G_{22}, G_{11}, G_{33}]$ in the axis of orthotropy.
If $\tau$ designates the shearing stress acting on $S$,

$$\tau = \alpha[G \cdot \varphi\nabla + G \cdot\ (e_1 \wedge \mathbf{0}_S M)]$$

with $\tau \cdot n = 0$ on $\partial S$.
There is no normal stress on $S$, and the resultant force on $S$ is equal to zero as can be demonstrated in the isotropic case.
Torque is obtained by

$$M = \int_S \mathbf{0}_S M \wedge \tau\ \mathrm{d}S = (e_1 \cdot M) e_1$$

Indeed

$$(e_1 \cdot \mathbf{0}_S M \cdot \tau) = \tau \cdot (e_1 \wedge \mathbf{0}_S M) = \tau \cdot \left(\frac{1}{\alpha} G^{-1} \cdot \tau - \varphi\nabla\right)$$

$$(\varphi\tau) \cdot \nabla = \varphi(\tau \cdot \nabla) + \varphi\nabla \cdot \tau = \varphi\nabla \cdot \tau$$

so

$$M = e_1 \cdot \int_S \left[\tau \cdot \frac{1}{\alpha} G^{-1} \cdot \tau - (\varphi\tau) \cdot \nabla\right] \mathrm{d}S$$

with

$$\int_S (\varphi\tau) \cdot \nabla\, \mathrm{d}S = \int_{\partial S} \varphi\tau \cdot n\, \mathrm{d}\ell = 0$$

Finally, we can put

$$M = \langle GJ \rangle\ \alpha e_1 = \Gamma e_1$$

where $\langle GJ \rangle$ is the torsional stiffness with

$$\langle GJ \rangle = \int_S (\varphi\nabla + e_1 \wedge \mathbf{0}_S M) \cdot G \cdot (\varphi\nabla + e_1 \wedge \mathbf{0}_S M)\ \mathrm{d}S$$

or also using Green-Riemann's theorem after some calculations,

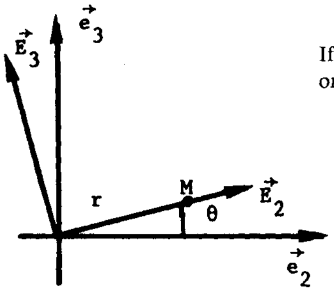
$$\langle GJ \rangle = \int_S (\mathbf{0}_S M \cdot G \cdot \mathbf{0}_S M - \varphi\nabla \cdot G \cdot \varphi\nabla)\ \mathrm{d}S$$

or

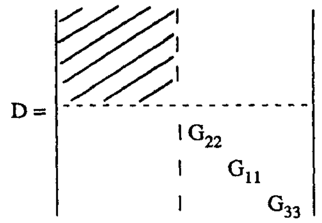
$$\langle GJ \rangle = \int_S [G_{33}x_2(x_2 + \varphi,_3) - G_{22}x_3(\varphi,_2 - x_3)]\ \mathrm{d}S$$

## Case of a Circular Shaft with Radius $R$

This problem was resolved using polar coordinates:




If matrix of elasticity in the axis of orthotropy is D.




In the axis $e_1$, $E_2$, and $E_3$, we have classically

$$D^* = TDT^T$$

with

$$T = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C^2 & S^2 & 0 & 2CS & 0 \\ 0 & S^2 & C^2 & 0 & -2CS & 0 \\ \hline 0 & 0 & 0 & C & 0 & S \\ 0 & -SC & SC & 0 & 1 & 0 \\ 0 & 0 & 0 & -S & 0 & C \end{array}\right]$$

$$(C = \cos\theta,\ S = \sin\theta)$$

The right corner of $D^*$ gives the new matrix $G^*(\theta)$.

$$G^*(\theta) = \left[\begin{array}{c|c|c} G_{22}C^2 + G_{33}S^2 & 0 & (G_{33} - G_{22})CS \\ \hline 0 & G_{11} & 0 \\ \hline (G_{33} - G_{22})CS & 0 & G_{22}S^2 + G_{33}C^2 \end{array}\right]$$

The problem to be handled becomes as follows: Find $\varphi(r,\theta)$ with a period $2\pi$ so that

$$(G_{22} + G_{33})/2\ [(r\ \varphi,_r),_r + (1/r)\varphi,_{\theta\theta}]$$
$$+ (G_{22} + G_{33})/2\{[(r\varphi,_r),_r - (1/r)\varphi,_{\theta\theta} - 2\varphi,_r] \cos 2\theta$$
$$- 2[\varphi,_r - (\varphi/r)],_\theta \sin 2\theta\} = 0$$

on $S$, and

$$(G_{22} + G_{33})\, \varphi,_r|_{r=R} + (G_{22} - G_{33})\{\varphi,_r|_{r=R} \cos 2\theta$$

$$- [R + (1/R)\varphi,_\theta]|_{r=R} \sin 2\theta\} = 0$$

on $\partial S$.

The solution sought is of the form

$$\varphi = \phi(r)\psi(\theta)$$

with $\psi(\theta),_{\theta\theta} = -4\psi(\theta)$, and after some trials, we adopt

$$\psi(\theta) = \sin 2\theta$$

and $\phi(r)$ as to verify

$$(r\phi,_r),_r - 4\phi/r = 0$$

$$(r\phi,_r),_r + 8\phi/r = -6\phi,_r = 0$$

so $\phi = Cr^2$.

$C$ is calculated using the boundary condition on $\partial S$:

$$C = (G_{22} - G_{33})/2(G_{22} + G_{33})$$

Finally,

$$\varphi = [(G_{22} - G_{33})/2(G_{22} + G_{33})]r^2 \sin 2\theta$$

and after some easy calculations,

$$\langle GJ\rangle = SR^2 \frac{1}{(1/G_{22}) + (1/G_{33})} = I_0\, 2[G_{22}\, G_{33}/(G_{22} + G_{33})]$$

## Conclusion

We found a free warping function of a beam of circular section acted upon by a torque in the direction of one of the axes of orthotropy. It occurs two shearing modulus. The result is just as it is an interesting one but may also be used for the experimental determination of shearing modulus making three tests of torsion in the three directions of orthotropy.

So, calling $X$, $Y$, $Z$ the three unknown modulus and $A$, $B$, $C$ $(\theta/2\alpha I_0)$ the three experimental results, it would be necessary to solve the nonlinear system

$$XY/(X+Y) = A, \qquad YZ/(Y+Z) = B$$

$$ZX/(Z+X) = C$$

or with the solution if

$$X = 2ABC/[B(A+C) - AC]$$

and $Y$ and $Z$ by cyclic permutation. Then, $X$, $Y$, $Z$ could be obtained.

The result may also be used for testing results of the finite element program on warping functions.

## References

[1]Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1956, Chap. 4.

[2]Vlasov, V. Z., *Thin Walled Elastic Beams*, English Translation, National Science Foundation, Washington, D.C., Oldbourne-Press, London, 1961.

[3]Courbon, J., *Résistance des matériaux*, Vol. 1, Dunod, Paris, 1964.

[4]Timoshenko, S., and Gere, S. P., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961, Chap. 5.

[5]Dubigeon, S., and Kim, C. B., "A Finite Element for the Study of Coupled Bending Prevented Torsion of a Straight Beam," *Journal of Sound and Vibration*, Vol. 81, No. 2, 1982, pp. 255–270.

# Optimal Shape Design of Lattice Structures for Accuracy

Hiroshi Furuya*
*Nagoya University, Nagoya 464-01, Japan*

## Introduction

THE purpose of this study is to present the optimal configuration of lattice space structures taking into account statistical sensitivity to shape distortions of the structures due to random member length errors. To design lattice space antennas, the length of the radio wave used for communication requires an extremely high surface accuracy. For accurate lattice structures, a new design concept is necessary to reduce structural errors. Sensitivity analyses on structural distortions provide significant information for designing tolerance errors of the structural elements and predicting the feasibility of fabricating accurate structures.

One must inevitably take into account the effects of random member length errors on the structural distortions for space structures consisting of a large number of members. Some studies based on approximate continuum analysis of the structures[1] and multiple deterministic structural analysis of the truss (the Monte Carlo method) using the finite element analysis[2] have been attempted to estimate the structural errors. Furthermore, for actively controllable structures on their deformation, the optimal locations of actuators have been studies to correct the shape distortions.[3]

## Statistical Sensitivity Analysis

In the present analysis it is assumed that structural errors can be regarded as stochastic ones and that the errors can be represented by member length tolerances. Formulations for statistical analysis on the accuracy from Ref. 4 are used herein. The formulas have been derived for expected value and variance of the structural error based on the assumption of zero-mean errors.

### Structural Deviations

A mathematical model of a three-dimensional lattice structure is defined as an assembly of straight members jointed at various nodal points. Thus, by assuming small deformation, the relationship between the member deformation vector and the corresponding nodal displacement vector can be written by the following in the matrix form:

$$dl = Adx \tag{1}$$

The equations of equilibrium can be derived from the variational principle by assuming small deformation. Thus, the equations of equilibrium at the node can be derived as follows:

$$A^TKAdx = A^TKds \tag{2}$$

where

$$K = \mathrm{diag}(E_iA_i/l_i), \qquad (i = 1, 2, \ldots, m) \tag{3}$$

When the lattice structure is in free boundary conditions, the rank of the matrix $(A^TKA)$ is not full because the lattice structure with free boundary has rigid mode nodal displacement. Therefore, by suppressing the rigid mode deformation,

Received July 15, 1991; revision received Feb. 14, 1992; accepted for publication Feb. 17, 1992. 

*Assistant Professor, Structures Engineering, Department of Aeronautical Engineering.